SIMPLEX METHOD

Simplex Method

- The simplex method is an iterative procedure.
- Beginning at a vertex of the feasible region S, each iteration brings us to another vertex of S with an *improved* value of the objective function.
- The iteration ends when the optimal solution is reached.

Standard Maximization Problems in Standard Form

A linear programming problem is said to be a standard maximization problem in standard form if its mathematical model is of the following form:

Maximize the objective function

$$Z_{\max} = P = c_1 x_1 + c_2 x_2 + ... + c_n x_n$$

Subject to problem constraints of the form

$$a_1 x_1 + a_2 x_2 + ... + a_n x_n \le b \qquad ,b \ge 0$$

With non-negative constraints

$$x_1, x_2, ..., x_n \ge 0$$

Slack Variables

A mathematical representation of surplus resources. In real life problems, it’s unlikely that all resources will be used completely, so there usually are unused resources.

- Slack variables represent the unused resources between the left-hand side and right-hand side of each inequality.

Slack and Surplus Variables

- A linear program in which all the variables are non-negative and all the constraints are equalities is said to be in standard form.
- Standard form is attained by adding slack variables to "less than or equal to" constraints, and by subtracting surplus variables from "greater than or equal to" constraints.
- Slack and surplus variables represent the difference between the left and right sides of the constraints.
- Slack and surplus variables have objective function coefficients equal to 0.

Slack Variables (for $\leq$ constraints)

Example 1 in Standard Form

$$\begin{array}{ll} \text{Max} & 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3 \\ \text{s.t.} & x_1 + 3x_2 + s_1 = 6 \\ & 2x_1 + 3x_2 + s_2 = 19 \\ & x_1 + x_2 + s_3 = 8 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{array}$$

s_1, s_2, and s_3 are slack variables

Surplus Variables

Example 2 in Standard Form

$$
\begin{array}{ll}
\text{Min} & 5x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 \\
\text{s.t.} & 2x_1 + 5x_2 - s_1 = 10 \\
& 4x_1 - x_2 - s_2 = 12 \\
& x_1 + x_2 - s_3 = 4 \\
& x_1, x_2, s_1, s_2, s_3 \geq 0
\end{array}
$$

s_1 , s_2 , and s_3 are surplus variables

Basic feasible solutions

- A *basic solution* to a system of m linear equations in n unknowns ($n \geq m$) is obtained by setting $n - m$ variables to 0 and solving the resulting system to get the values of the other m variables.

- The variables set to 0 are called *non-basic*; the variables obtained by solving the system are called *basic*.

- A basic solution is called *feasible* if all its (basic) variables are nonnegative.

Example

$$x + y + u = 4$$
$$x + 3y + v = 6$$

(0, 0, 4, 6) is basic feasible solution

(x, y are non-basic; u, v are basic)

Simplex Example-1

Max $z = 3x_1 + 2x_2$

st: $x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

Simplex Example-2

$$\text{Max } z = 4x_1 + 10x_2$$

st:

$$2x_1 + x_2 \leq 10$$
$$2x_1 + 5x_2 \leq 20$$
$$2x_1 + 3x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

Simplex Example-3

$$\text{Min } z = x_1 - 3x_2 + 2x_3$$

$$\begin{aligned}
\text{st:}\quad & 3x_1 - x_2 + 2x_3 \leq 7 \\
& -2x_1 + 4x_2 \leq 12 \\
& -4x_1 + 3x_2 + 8x_3 \leq 10 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

DUALITY THEORY

The Essence of Duality

- One of the most important discoveries in the early development of linear programming was the concept of duality.
- Every linear programming problem is associated with another linear programming problem called the **dual**.
- The relationships between the dual problem and the original problem (called the **primal**) prove to be extremely useful in a variety of ways.

Duality in Linear Programming

- Every linear program has another linear program associated with it i.e., its **‘dual’**
- The dual complements the original linear program, the **‘primal’**

- The initial problem is known as **primal** and the transpose problem obtained by transposing row into columns is known as **dual**.
- Each linear programming problem can analyzed in two different ways but would have equivalent solution.

Primal and Dual Problems

Primal Problem	Dual Problem
Max $Z = \sum_{j=1}^{n} c_j x_j$	Min $W = \sum_{i=1}^{m} b_i y_i$
s.t. $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$	s.t. $\sum_{i=1}^{m} a_{ij} y_i \geq c_j$
for $i = 1, 2, \ldots, m.$	for $j = 1, 2, \ldots, n.$
$x_j \geq 0,$ for $j = 1, 2, \ldots, n.$	$y_i \geq 0,$ for $i = 1, 2, \ldots, m.$

The dual problem uses exactly the same parameters as the primal problem, but in different location.

Example

Primal Problem in Algebraic Form

Max $Z = 3x_1 + 5x_2$

s.t.

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0, x_2 \geq 0$$

Dual Problem in Algebraic Form

Min $W = 4y_1 + 12y_2 + 18y_3$

s.t.

$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

Primal-dual table for linear programming

		Primal Problem				
		Coefficient of:				Right Side
		x_1	x_2	$\cdots$	x_n	
Dual Problem — Coefficient of:	y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\leq b_1$
	y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\leq b_2$
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
	y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\leq b_m$
Right Side		VI	VI	$\cdots$	VI	
		c_1	c_2	$\cdots$	c_n	

Coefficients for Objective Function (Maximize)

Coefficients for Objective Function (Minimize)

Relationships between Primal and Dual Problems

One Problem		Other Problem
Constraint i	$\longleftrightarrow$	Variable i
Objective function	$\longleftrightarrow$	Right sides

	Minimization		Maximization	
Variables	≥ 0	$\longleftrightarrow$	$\leq$	Constraints
	≤ 0	$\longleftrightarrow$	$\geq$	
	Unrestricted	$\longleftrightarrow$	$=$	
Constraints	$\geq$	$\longleftrightarrow$	≥ 0	Variables
	$\leq$	$\longleftrightarrow$	≤ 0	
	$=$	$\longleftrightarrow$	Unrestricted	

- The feasible solutions for a dual problem are those that satisfy the condition of optimality for its primal problem.
- A maximum value of Z in a primal problem equals the minimum value of W in the dual problem.
- Any pair of primal and dual problems can be converted into each other.
- The dual of a dual problem is always the primal problem.

Question 1: Consider the following problem.

$$\text{Maximize} \qquad Z = 2x_1 + 4x_2 + 3x_3$$

subject to

$$3x_1 + 4x_2 + 2x_3 \le 60$$

$$2x_1 + x_2 + 2x_3 \le 40$$

$$x_1 + 3x_2 + 2x_3 \le 80$$

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Write the dual of the above LPP.

Question 2: Consider the following problem.

Minimize $Z = 3x_1 - 2x_2 + 4x_3$

subject to

$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$7x_1 - 2x_2 - x_3 \leq 10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_1 + 7x_2 - 2x_3 \geq 2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Write the dual of the above LPP.

Question 3:

Consider the following problem.

Maximize $Z = -x_1 - 2x_2 - x_3$

subject to

$$x_1 + x_2 + 2x_3 \le 12$$

$$x_1 + x_2 - x_3 \le 1$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(a) Construct the dual problem.

(b) Use duality theory to show that the optimal solution for the primal problem has $Z \le 0$.

Question 4:

For each of the following linear programming models, give your recommendation on which is the more efficient way (probably) to obtain an optimal solution: by applying the simplex method directly to this primal problem or by applying the simplex method directly to the dual problem instead. Explain.

(a) Maximize $Z = 10x_1 - 4x_2 + 7x_3$

subject to

$$3x_1 - x_2 + 2x_3 \le 25$$
$$x_1 - 2x_2 + 3x_3 \le 25$$
$$5x_1 + x_2 + 2x_3 \le 40$$
$$x_1 + x_2 + x_3 \le 90$$
$$2x_1 - x_2 + x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(b) Maximize

$$Z = 2x_1 + 5x_2 + 3x_3 + 4x_4 + x_5$$

subject to

$$x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 \le 6$$
$$4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 \le 15$$

and

$$x_j \ge 0, \quad for\ j = 1, 2, 3, 4, 5.$$

SENSITIVITY ANALYSIS

Sensitivity Analysis

- Sensitivity analysis allows us to determine how “sensitive” the optimal solution is to changes in data values.
- How does the value of the optimum solution change when coefficients in the obj. function or constraints change?

This includes analyzing changes in:

i) An Objective Function Coefficient (OFC)

ii) A Right Hand Side (RHS) value of a constraint

Graphical Sensitivity Analysis

We can use the graph of an LP to see what happens when:

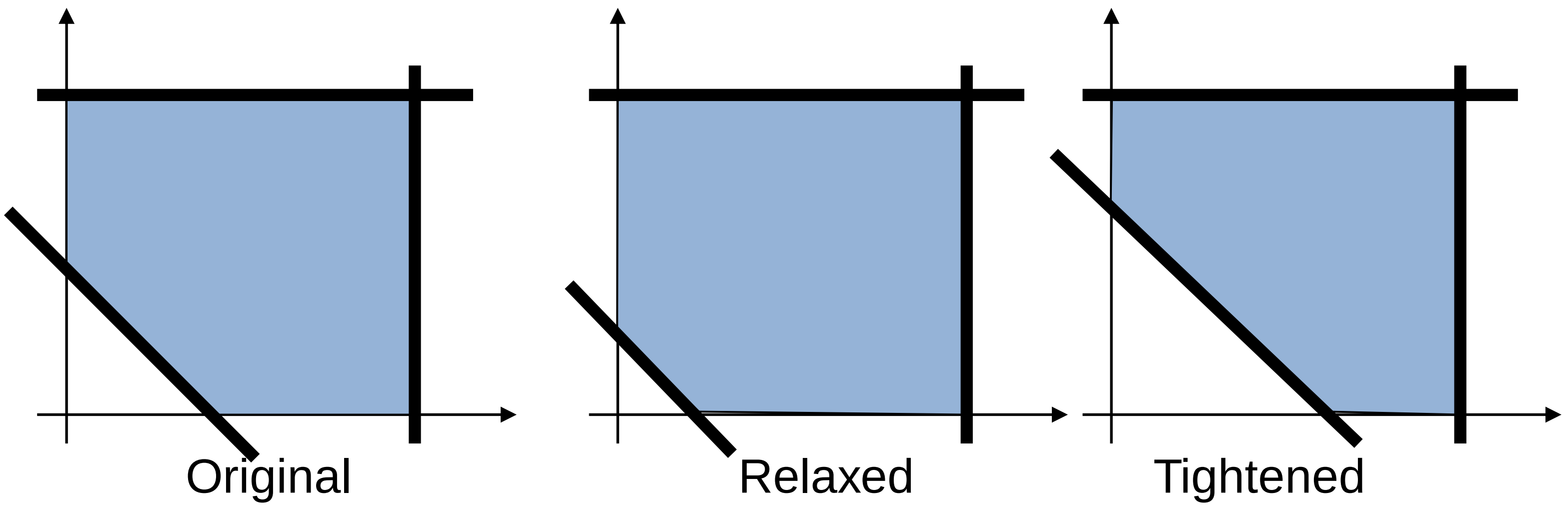
i) An OFC changes

ii) A RHS changes

Shadow price: The change in optimum value of obj. function per unit change in the constraint right hand value.

Changing Constraints

- Relaxing constraints:
  - Optimal value same or better
- Tightening constraints:
  - Optimal value same or worse



Ex- Flair Furniture Company

- The Flair Company produces inexpensive tables and chairs. Each table takes 3 hours of carpentry and 2 hrs in painting shop. Each chair requires 4 hours in carpentry and 1 hours in painting shop. The current production period, 2400 hrs of carpentry time is available and 1000 hrs in painting time are available. Each table sold yields a profit of $7; each chair produced is sold for a $5 profit. The company has to produce atleast 100 tables and atmost 450 chairs.
- Flair Furniture’s problem is to determine the best possible combination of tables and chairs to manufacturer in order to reach the maximum profit. The firm would like this production mix situation formulated as an LP Problem.

Flair Furniture Problem

$$\text{Max } Z = 7T + 5C \qquad \text{(profit)}$$

Subject to the constraints:

$$3T + 4C \leq 2400 \qquad \text{(carpentry hrs)}$$

$$2T + 1C \leq 1000 \qquad \text{(painting hrs)}$$

$$C \leq 450 \qquad \text{(max \# chairs)}$$

$$T \geq 100 \qquad \text{(min \# tables)}$$

$$T, C \geq 0 \qquad \text{(non-negativity)}$$

Objective Function Coefficient (OFC) Changes

What if the profit contribution for tables changed from \$7 to \$8 per table?

$$\text{Max } Z = \overset{8}{\cancel{7}}\ T + 5\ C \qquad\qquad \text{(profit)}$$

Clearly profit goes up, but would we want to make more tables and less chairs?
(i.e. Does the optimal solution change?)

Characteristics of OFC Changes

- There is no effect on the feasible region
- The slope of the Iso-profit line changes
- If the slope changes enough, a different corner point will become optimal

Original

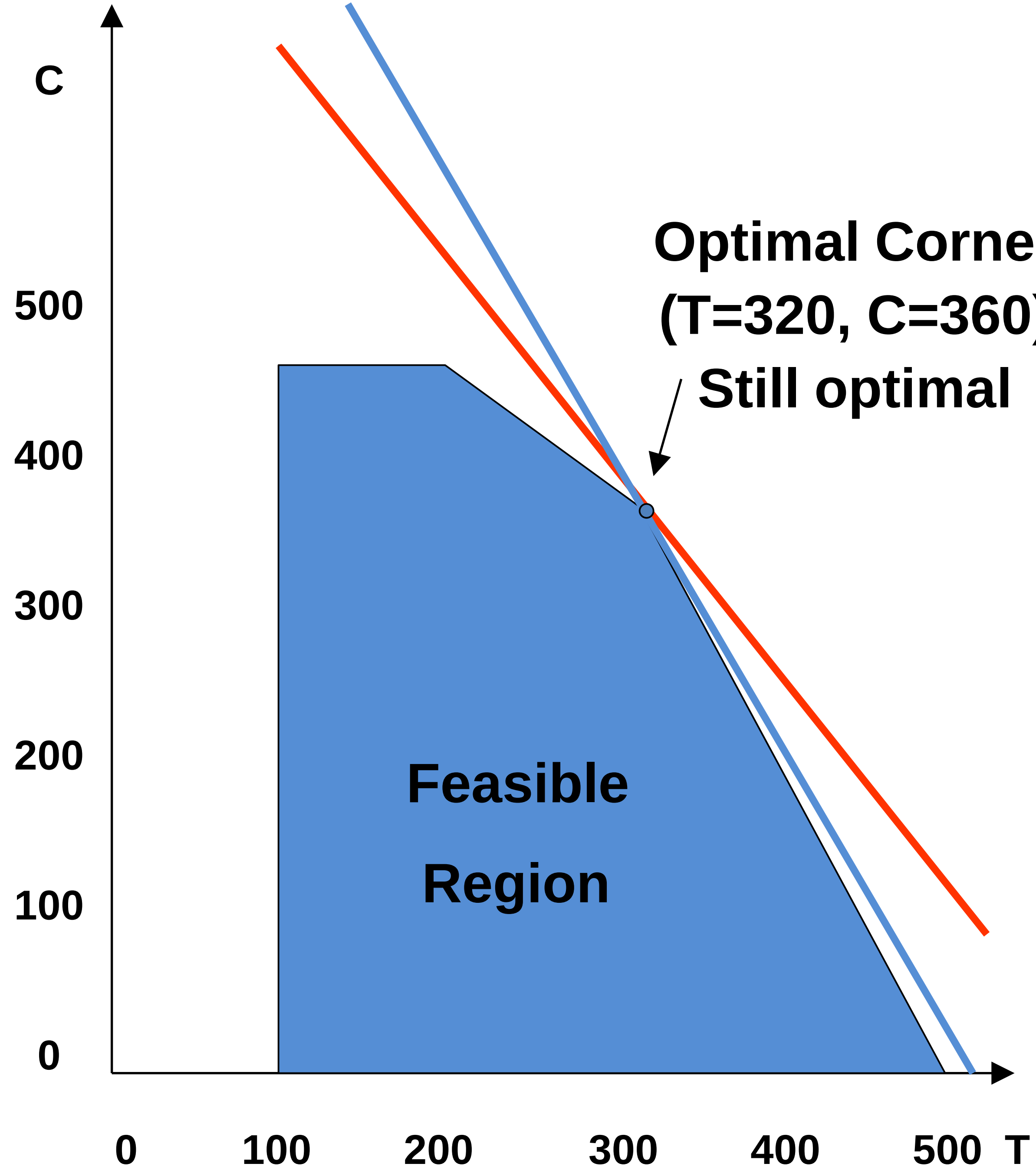
<u>Objective Function</u>

7T + 5 C = $4040

Revised

<u>Objective Function</u>

8T + 5 C = $4360



What if the OFC became higher? Or lower?

11T + 5C = $5500

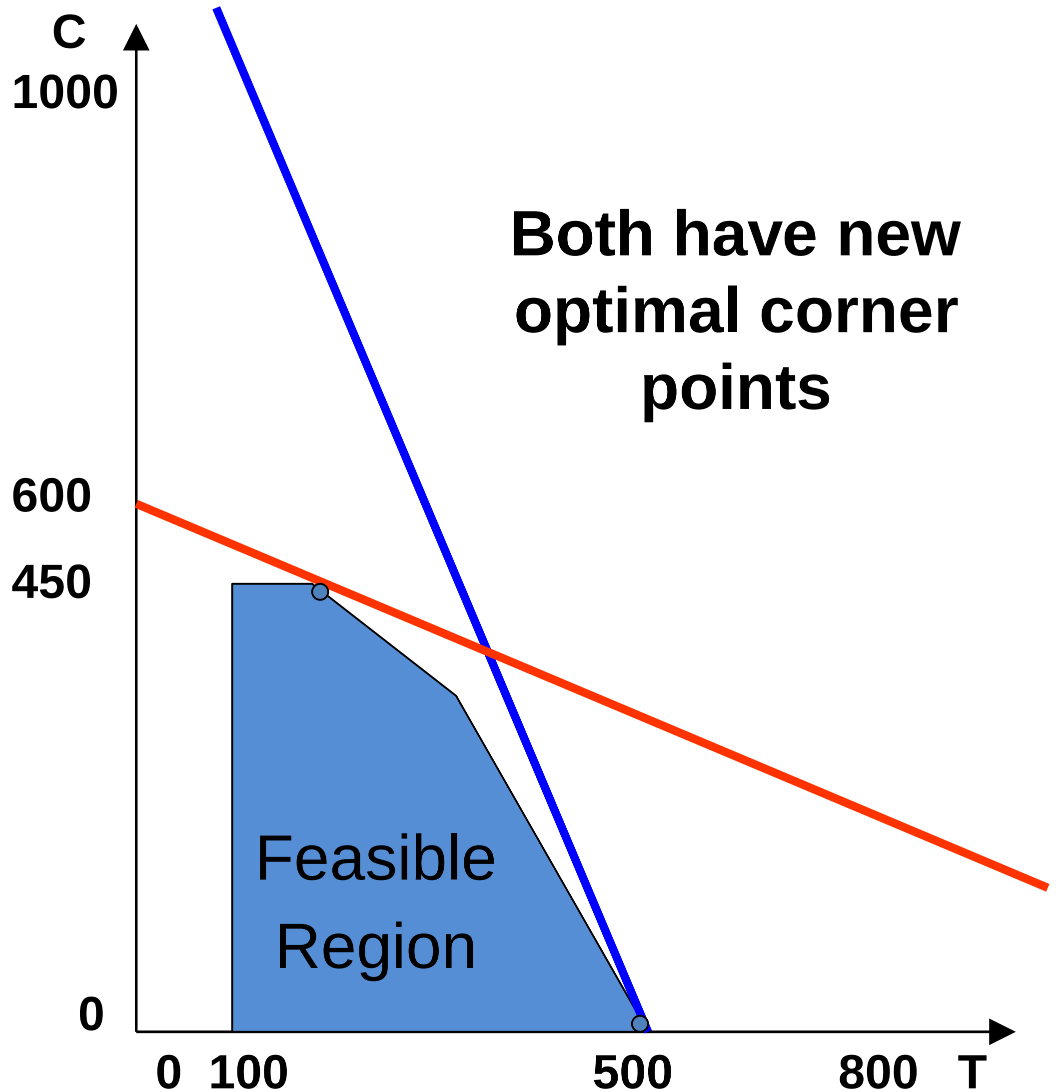
Optimal Solution

(T=500, C=0)

3T + 5C = $2850

Optimal Solution

(T=200, C=450)



- There is a range for each OFC where the current optimal corner point remains optimal.
- OFC range is given by:

$$-2 \le -C_1/C_2 \le -3/4$$

$$3/4 \le C_1/C_2 \le 2$$

(C_1, C_2 are coefficient of decision variable)

- If the OFC changes beyond that range a new corner point becomes optimal.

Right Hand Side (RHS) Changes

What if painting hours available changed from 1000 to 1300?

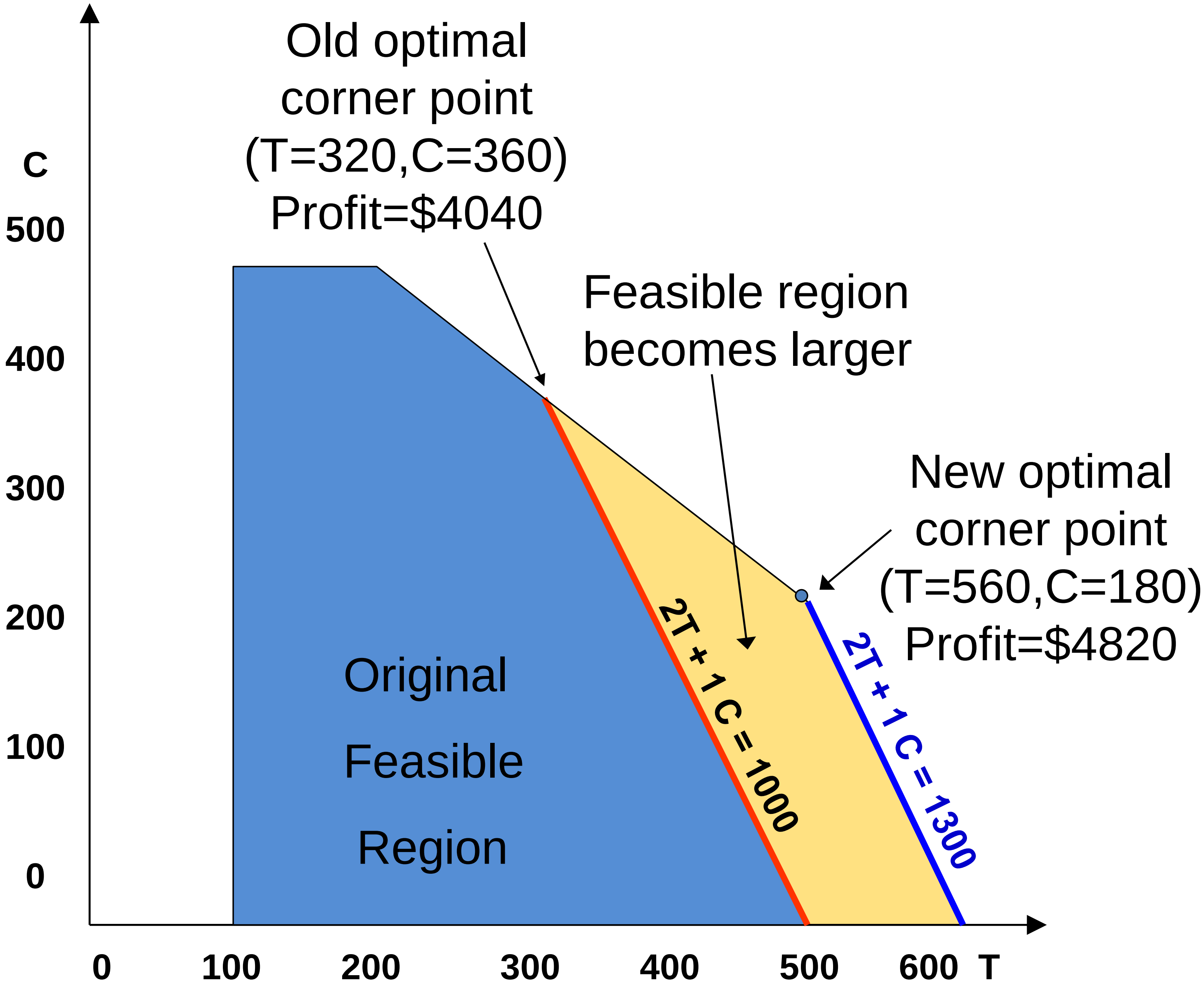
1300

$$2T + 1C \leq \cancel{1000} \quad \text{(painting hrs)}$$

This increase in resources could allow us to increase production and profit.

Characteristics of RHS Changes

- The constraint line shifts, which could change the feasible region
- Slope of constraint line does not change
- Corner point locations can change
- The optimal solution can change



Old optimal corner point (T=320,C=360) Profit=$4040
Feasible region becomes larger
New optimal corner point (T=560,C=180) Profit=$4820
Original Feasible Region
2T + 1 C = 1000
2T + 1 C = 1300
C
500
400
300
200
100
0
0
100
200
300
400
500
600
T

Effect on Objective Function Value

New profit	= \$4,820
Old profit	= \$4,040
Profit increase	= \$780 from 300 additional painting hours

\$2.60 in profit per hour of painting

- Each additional hour will increase profit by \$2.60
- Each hour lost will decrease profit by \$2.60

Shadow Price

- The change in the objective function value per one-unit increase in the RHS of the constraint.

Shadow Price:

$$SP = \frac{(Z_2 - Z_1)}{\Delta \text{Resource}} = \frac{(4820 - 4040)}{(1300 - 1000)} = 2.60 \text{ (\$)/hr}$$

Constraint RHS Changes

If the change in the RHS value is within the allowable range, then:

- The shadow price does not change
- The change in objective function value =

  (shadow price) x (RHS change)

- If the RHS change goes beyond the allowable range, then the shadow price will change.

Example

Q. A company manufactures two products, A and B. Product A requires 2 hrs on machine-1 and 1 hr on machine-2 where as product-B requires 1 hr on machine-1 and 2 hrs on machine-2. Both the machines are available for 8 hours daily. The profit contribution per unit of A and B are $ 300 and $ 200 respectively. Formulate the LPP to maximize profit?

Sensitivity Analysis calculation

a) What will happen if machine-1 availability will increase from 8hrs to 9hrs daily? (Hint: shadow price?)
b) What will happen if machine-2 availability will increase from 8hrs to 9hrs daily?
c) Based on above calculation which machines capacity should be increased?
d) Should we increase the availability of machine-1 at an additional cost of $ 50? What about machine-2?
e) Suppose the unit price for product A and product-B are changed to $ 350 and $ 250 respectively. Will the current optimal solution remain the same?
f) Suppose the unit profit of product-B is fixed at its current value C_2 = $ 200 then what is the range of the unit profit for product-A, C_1, that will keep the optimal solution unchanged?